Functional analysis
List 0

Zad 1. Let $x, y \in \mathbb{R}^n$ and $p > 1$. Prove *Hölder inequality* (1884):

$$\sum_{k=1}^{n} |x(k)y(k)| \leq \Big(\sum_{i=1}^{n} |x(k)|^p\Big)^{\frac{1}{p}} \Big(\sum_{i=1}^{n} |y(k)|^q\Big)^{\frac{1}{q}},$$

where $q \geq 1$ is such that $\frac{1}{p} + \frac{1}{q} = 1$. Moreover check that this inequality becomes equality if and only if

$$\exists_{\lambda \in \mathbb{R}} \ \forall_{k=1,\dots,n} \quad |x(k)|^p = \lambda |y(k)|^q.$$

Zad 2. Show that for $p \geq 1$ the function $\|x\|_p = (\sum_{k=1}^{n} |x(k)|^p)^{\frac{1}{p}}$ is a norm in $\mathbb{R}^n$.

Zad 3. Examin the limit $\lim_{p\to\infty} \|x\|_p$, $x \in \mathbb{R}^n$, and show that $\|x\|_\infty = \max_{1\leq k\leq n} |x(k)|$ is a norm in $\mathbb{R}^n$.

Zad 4. Show that the sequence $\{x_k\}_{k\in\mathbb{N}}$ in the space $(\mathbb{R}^n, \|\cdot\|_p)$, $p \in [1, \infty]$, is convergent to x_0 if and only if $\lim_{k\to\infty} x_k(i) = x_0(i)$ for all $i = 1, ..., n$.

Zad 5. Prove that $(\mathbb{R}^n, \|\cdot\|_p)$, $p \in [1, \infty]$, are Banach spaces.

Zad 6. Let $A : (\mathbb{R}^n, \|\cdot\|_p) \to (\mathbb{R}^m, \|\cdot\|_q)$ be a linear map with the corresponding matrix $[a_{ij}]_{\substack{1\leq i\leq m \\ 1\leq j\leq n}}$. Prove that, the norm of operator A

a) for $p = 1$ and $q = \infty$ is expressed by the formula $\|A\| = \max_{\substack{1\leq i\leq m \\ 1\leq j\leq n}} |a_{ij}|$,

b) for $p = q = \infty$ is expressed by the formula $\|A\| = \max_{1\leq i\leq m} \sum_{j=1}^{n} |a_{ij}|$,

c) for $p = q = 2$ is limited by the following inequalites

$$\max_{i,j} |a_{ij}| \leq \|A\| \leq \sqrt{\sum_{i,j} |a_{ij}|^2}.$$

Zad 7. Compute the norms of operators acting on the euclidean space $(\mathbb{R}^2, \|\cdot\|_2)$

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \qquad C = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$